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On density of state of quantized Willmore surface—a way to a quantized extrinsic string in $\mathbb{R}^3$

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Abstract. An elastica has been recently quantized with the Bernoulli–Euler functional in two-dimensional space using the modified Korteweg–de Vries hierarchy. In this paper a Willmore surface is quantized, or equivalently a surface with the Polyakov extrinsic curvature action, using the modified Novikov–Veselov (MNV) equation. In other words, it is shown that the density of states of the partition function for the quantized Willmore surface is expressed by the volume of a subspace of the moduli of the MNV equation.

1. Introduction

In a series of works [1–7], the correspondence between an immersed object and the Dirac operator confined there has been considered. The Dirac operator confined in an immersed object is uniquely determined by the procedure which was proposed previously [1–4] and can be regarded as the representation matrix of the symmetry of the immersed object [1–7]. It had been studied mainly on an elastica in a plane [1–6], which is a model of a thin elastic rod. Then it was shown that the Dirac operator confined in an elastica can be identified with the Lax operator of the modified Korteweg–de Vries (MKdV) equation (1.5) while the mathematical deformation of the elastica obeys the MKdV hierarchy [6, 7]. By investigating other quantum equations [8–9], it is conjectured that such correspondence between the Dirac operator and its geometry can be extended to higher-dimensional immersed objects [2–4].

Recently Konopelchenko [10, 11] discovered that a conformal surface $\mathcal{S}$ immersed in three-dimensional flat space $\mathbb{R}^3$ obeys the Dirac equation, which we will call the Konopelchenko–Kenmotsu–Weierstrass–Enneper (KKWE) [10–15] equation here

$$\partial f_1 = V f_2 \qquad \bar{\partial} f_2 = -V f_1 \tag{1.1}$$

where

$$V := \tfrac{1}{2}\sqrt{\rho} H \tag{1.2}$$

H is the mean curvature of the surface $\mathcal{S}$ parametrized by complex z and ρ is the conformal metric induced from $\mathbb{R}^3$. The KKWE equation completely exhibits immersed geometry as the old Weierstrass–Enneper equation expresses the minimal surface [10–15]. In [16] we show that it is identified with the Dirac operator confined in the surface $\mathcal{S}$ using my confinement procedure. By quantizing the Dirac field we find that the quantized symmetry of the Dirac operator is also in agreement with the symmetry of the surface itself [17]. In other words, this KKWE equation is the equation which was conjectured before [2, 3] and had been searched for. Although for a more general surface, which is not conformal,

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the KKWE type equation was discovered by Burgress and Jensen [18] following from prescriptions [1], their equation is not easy to deal with and we could not find meaningful results. However, the KKWE equation is very useful in investigating the immersed object and in terms of (1.1), Konopelchenko, Taimanov and other Russian groups find non-trivial results related to the immersed surface [10–14, 19, 20].

By physical investigation of the KKWE equation and its quantized version, the Willmore functional [21, 22] and the modified Novikov–Veselov (MNV) equation naturally appear [10–14, 17, 19, 20]. The Willmore functional is given as

$$W = \int_{\mathcal{S}} \mathrm{dvol}\, H^2 \tag{1.3}$$

where 'dvol' is a volume form of the surface $\mathcal{S}$. The harmonic map associated with this functional has been studied in differential geometry [21, 22].

On the other hand, Polyakov introduced an extrinsic curvature action in string theory and the theory of two-dimensional gravity from renormalizability [23]. However, his action is just the Willmore functional (1.3). Thus, his programme was recently investigated by Carroll and Konopelchenko [19] and Grinevich and Schmidt [20] using the KKWE equation (1.1). The main aim of this paper is to quantize the Willmore surface but it must be emphasized that this involves the study of the quantization of the Polyakov extrinsic curvature action.

It should be noted that the elastica problem of $\mathbb{R}^2$ has a very similar structure to that of the Willmore surface problem of $\mathbb{R}^3$ [10–14]. Corresponding to the Willmore functional (1.3), there is the Bernoulli–Euler functional for an elastica in $\mathbb{R}^2$ [24]

$$E = \int \mathrm{d}q^1\, k^2 \tag{1.4}$$

where k is a curvature of the elastica [24]. Mathematically speaking, an elastica is a non-stretching curve immersed in a higher-dimensional manifold, e.g. $\mathbb{R}^2$, realized as a minimal point of the energy functional (1.4). For the case of the n-dimensional manifold $n > 3$, the Bernoulli–Euler functional is sometimes modified. Whereas the Willmore surface is related to the MNV equation, the elastica is related to the MKdV equation [1–7, 25–27].

Recently we have quantized exactly the elastica of the Bernoulli–Euler functional (1.4) preserving its local length [25]. Then it was found that its moduli are completely represented by the MKdV equation and are closely related to the two-dimensional quantum gravity [28–30]. The quantized elastica obeys the MKdV hierarchy and at a critical point, the Painlevé equation of the first kind appears [25] while in the quantized two-dimensional gravity which is defined at a critical point of the discrete tiling model, the Painlevé equation of the first kind appears with the Korteweg–de Vries (KdV) hierarchy [28–30].

In this paper, instead of preserving the local length, we require that the surface retains its conformal structure and we quantize the Willmore functional. Then, it will be shown that the MNV equation appears as the virtual quantized motion of a Willmore surface in the path integral.

The organization of this paper is as follows. Section 2 reviews the argument of the quantized elastica following that in [25]. In section 3, we quantize the Willmore surface and then the density of states of the Willmore functional is given as a volume determined by the MNV equation. Section 4 discusses these results.

2. Quantization of elastica

Whereas studies on the Willmore functional are current and proceeding now, those of the elastica have a long history. The problem of an elastica in $\mathbb{R}^2$ was proposed in the

17th century. Its static properties were investigated by Euler and Bernoulli's family in the 17th and 18th centuries and were completely classified by Euler [24]. However, its dynamics are very difficult and were partially studied related to the MKdV equation [6, 7]. The energy functional of the dynamics of the elastica includes a kinematic term added to (1.4). Since the kinematic term weakly prevents the integrable properties of the elastica's development through physical time, an elastica is not governed in general by the MKdV equation but approximately obeys the MKdV equation [6, 7]. However, the mathematical deformation of the elastica with only the potential energy or the Bernoulli–Euler functional (1.4) has a structure of MKdV hierarchy [6, 25–27]. When one deals with a mathematical deformation, we encounter another 'time' as a deformation parameter. Readers should note that in [25–27] and in this paper, the word 'time' is used in this sense but it should not be confused with a physical (real) time [6,7]. The MKdV equation and hierarchy is well studied, at least formally, using the Jimbo–Miwa theory [31] and the mathematical deformation of a curve has been investigated by Goldstein and Petrich [26, 27]. Based on well established theories, we could quantize the elastica using the MKdV hierarchy and inspect it as in [25].

On the other hand, studies on the Willmore surface and the MNV equation have only a short history and thus are not well established. Accordingly in this article, we will quantize the Willmore surface along the lines of the quantization procedure of the elastica. Thus, before we describe the Willmore surface, in this section, we will review the calculation scheme of the partition function of elastica which was discovered in [25] and is interpreted as quantization of elastica.

We denote by $\mathcal{C}$ a shape of an elastica (an ideal thin elastic rod or a real one-dimensional curve) immersed in a complex plane $\mathbb{C}$ and by $X(s)$ its affine vector [6]

$$S^1 \ni s \mapsto X(s) \in \mathcal{C} \subset \mathbb{C} \qquad X(s+L) = X(s) \tag{2.1}$$

where L is the length of the elastica. We fix the metric of the curve $\mathcal{C}$ induced from the natural metric of $\mathbb{C}$; $\mathrm{d}s = (\mathrm{d}X\,\mathrm{d}\bar{X})^{1/2}$. The Frenet–Serret relations are expressed as [6, 25–27]

$$\psi_0 := \exp(\mathrm{i}\phi/2) = \sqrt{\partial_s X} \tag{2.2}$$

$$\begin{pmatrix} \partial_s & v \\ v & -\partial_s \end{pmatrix} \begin{pmatrix} \psi_0 \\ \mathrm{i}\psi_0 \end{pmatrix} = 0 \qquad v := \frac{1}{2}k := \frac{1}{2}\partial_s\phi \tag{2.3}$$

where ϕ is a real valued function of s and k is the curvature of the curve $\mathcal{C}$, $\phi(s+L) = \phi(s)$ and $k(s+L) = k(s)$.

The energy functional of the elastica, which we will call the Bernoulli–Euler functional here [24], is given as

$$E = \int_0^L \mathrm{d}s\, k^2 = 4\int_0^L \mathrm{d}s\, v^2. \tag{2.4}$$

An elastica is defined as a non-stretching curve realized as a stationary point of the energy functional (2.4). In other words, an elastica is a model of a thin elastic rod; the elastic force comes from its thickness and elasticity but one can assume that its thickness is negligible so that it is interpreted as a mathematical curve [24]. Here readers should not mix an elastica and a 'string' in string theory up by mistake because (2.4) does differ from the Nambu–Goto action and ordinary Polyakov action of an ordinary string in string theory [23]; a 'string' in string theory is a relativistic curve and is a surface including its trajectory while string in the violin resembles an elastica rather than a 'string' even though a primary text book of string theory might illustrate a mode of a 'string' using the oscillation of a violin. From this point, we will only use the term 'string' in the sense of string theory with discriminating from elastica.

We assume that the elastica is closed and preserves its local infinitesimal length in the quantization process. It does not stretch. The partition function of the elastica is given as [25]

$$\mathcal{Z} = \int DX \exp\left(-\beta \int_0^L \mathrm{d}s\, k^2 \right) \tag{2.5}$$

where β is a quantization parameter; β can be regarded as the inverse of Planck constant $\hbar$ for Euclidean quantum mechanics and also as the inverse of temperature for problems of statistical mechanics.

Here we note that there are trivial symmetries on this system and that the partition function (2.5) naturally diverges [25]. For an affine transformation (translation and rotation) and a reparametrization of S^1, $X(s) \to X_0 + \mathrm{e}^{\mathrm{i}\phi_0} X(s + s_0)$, ($X_0$, ϕ_0 and s_0 are constants of s) and the curvature k and the energy functional (2.4) are invariant. In other words, these are gauge freedoms and the energy functional (2.4) has infinitely degenerate states. By philosophies of the functional integral in which we must sum over all possible states, $\mathcal{Z}$ includes the integration over $\mathbb{C}$ induced by the translation and naturally diverges. Hence we regularize it

$$\mathcal{Z}_{\mathrm{reg}} = \frac{\mathcal{Z}}{\mathrm{Vol(Aff)}L} \tag{2.6}$$

where Vol(Aff) and L is the volume of the trivial space in which the Bernoulli–Euler functional is invariant for the affine transformation and the reparametrization of the origin of S^1 [25]. As we show later, the measure in (2.6) can be regarded as the Haar measure of a group manifold which exhibits the symmetry of this quantized system, this regularization can be interpreted as the quotient space of a total group manifold by these transformation groups as normal subgroups. By the regularization, we can concentrate on classifying shape of the elastica itself. In other words, we choose the origin of s, a starting point and its tangential angle of the elastica in $\mathbb{C}$ as a representative of the operator domain for these transformation groups.

Next, we consider the condition of local length preservation. In the path integral, we must pay attention to the higher perturbations of ϵ to gain an exact result. Hence we assume that X is parametrized by a parameter t and that the difference between the perturbed affine vector X_ϵ and the unperturbed one X can be expressed by [6, 25–27]

$$X_\epsilon(s,t) := \mathrm{e}^{\epsilon\partial_t} X_{\mathrm{qcl}}(s,t) \qquad \epsilon\partial_t X = X_\epsilon - X + \mathcal{O}(\epsilon^2) \tag{2.7}$$

with the relation

$$-\partial_t X_{\mathrm{qcl}} = (u_1 + \mathrm{i}u_2)\exp(\mathrm{i}\phi_{\mathrm{qcl}}) \qquad u_1(L) = u_1(0) \qquad u_2(L) = u_2(0) \tag{2.8}$$

where u are real functions of s and t. This is virtual dynamics of the curve [6]. As well as the argument in [6, 25–27], due to the isometry condition, we require $[\partial_t, \partial_s] = 0$ for X. Then the isometry condition exactly preserves, $\mathrm{d}s \equiv \mathrm{d}s_\epsilon$ for $\mathrm{d}s_\epsilon := \sqrt{\partial_s \bar{X}_\epsilon \partial_s X_\epsilon}\,\mathrm{d}s$. Even though ϵ is constant, dependence of the variation upon the position s comes from the 'equation of motion' (2.8) and $u_a(s)$, $a = 1, 2$. Hence the deformation (2.7) contains non-trivial ones.

From $[\partial_t, \partial_s] = 0$, we have the relation [26, 27]

$$-\partial_t \exp(\mathrm{i}\phi_{\mathrm{qcl}}) = ((\partial_s u_1 - u_2 k_{\mathrm{qcl}}) + \mathrm{i}(\partial_s u_2 + u_1 k))\exp(\mathrm{i}\phi_{\mathrm{qcl}}). \tag{2.9}$$

Noting that ϕ and u are real valued, (2.9) is reduced to two coupled differential equations and by partially solving one of them, we obtain the 'equation of motion' of the deformation

$$\partial_s u_1 = k_{\mathrm{qcl}} u_2 \qquad u_1 = \int^s \mathrm{d}s\, u_2 k_{\mathrm{qcl}} =: \partial_s^{-1} u_2 k_{\mathrm{qcl}}$$
$$\partial_t k = -\Omega u_2. \tag{2.10}$$

Here ∂_s^{-1} is the pseudo-differential operator with a parameter $c \in \mathbb{R}$ as an integral constant and

$$\Omega := \partial_s^2 + \partial_s k_{\mathrm{qcl}} \partial_s^{-1} k_{\mathrm{qcl}} \tag{2.11}$$

is the Gel'fand–Dikii operator for the MKdV equation [26, 27].

In [6, 25], instead of the single deformation parameter, we use the infinite dimensional parameters $\boldsymbol{t} = (t_1, t_3, \ldots)$ and investigate the moduli space of the partition function. Then the minimal set of the virtual equations of motion, which are satisfied with certain physical requirements, is given as

$$\partial_{t_{2n+1}} k_{\mathrm{qcl}} = -\Omega^n \partial_s k_{\mathrm{qcl}} \qquad \partial_{t_{2n+3}} k_{\mathrm{qcl}} = \Omega \partial_{t_{2n+1}} k_{\mathrm{qcl}} \qquad (n = 1, 2, \ldots). \tag{2.12}$$

They are the MKdV hierarchy [27, 28]. As in [6], these relations (2.12) should be regarded as the Noether currents for the immersed object and t should be considered as the Schwinger proper times, in [25] it was shown that (2.12) means the quantum fluctuations and currents given by the quantized Noether theorem or the Ward–Takahashi identities.

However, by studying the moduli of the quantized elastica, the non-trivial deformation obeys the MKdV equation, which is obtained as $u = \partial_s k$ $(k = 2v)$ in (2.10)

$$\partial_t v + 6v^2 \partial_s v + \partial_s^3 v = 0 \tag{2.13}$$

because the solutions of the higher order equations belonging to the MKdV hierarchy are also satisfied with the MKdV equation.

Here it is a very remarkable fact that for the variation of t to obey the MKdV equation, the Bernoulli–Euler functional is invariant

$$\partial_t \int \mathrm{d}s\, v(s,t)^2 = \frac{1}{4} \partial_t E = 0 \tag{2.14}$$

because

$$\partial_t \int \mathrm{d}s\, v^2 = -\int \mathrm{d}s\, \partial_s \left(\frac{3}{2} v^4 + \frac{1}{2} (v \partial_s^2 v - (\partial_s v)^2 \right) = 0. \tag{2.15}$$

Since the MKdV problem is an initial value problem, for any regular shape of elastica satisfied by the boundary conditions, the 'time' t development of its curvature can be expressed by (2.13). In other words, for any given regular curve as an initial condition, there exists a family of solutions of the MKdV equation (2.13) as an orbit which contain the given curve. Due to the integrability and (2.14), during the motion of t, the energy functional does not change its value. Hence the trajectory of the deformation parameter t draws a curve in the functional space which has the same value of the energy functional (2.4). This reminds us of the fact that in group theory the character of a group is invariant among the elements belonging to the same conjugate class. In fact in Jimbo–Miwa theory, the solution space of the MKdV equation is acted by the affine Lie algebra $A_1^{(1)}$ [31].

Thus we can estimate the functional space for each functional value. In other words by investigating the subset of the moduli of the MKdV equation which is satisfied with the boundary conditions

$$v(0) = v(L) \qquad X(0) = X(L) \tag{2.16}$$

the measure of the functional integral (2.6) $\mathrm{d}\mu$ can be decomposed

$$\mathrm{d}\mu = \sum_E \mathrm{d}\mu_E. \tag{2.17}$$

So we denote by Ξ_E the set of these trajectories which occupy the same energy E.

Hence the partition function can be represented as

$$\mathcal{Z}_{\mathrm{reg}} = \int \mathrm{d}\mu \exp(-\beta E) = \sum_E \exp(-\beta E) \int_{\Xi_E} \mathrm{d}\mu_E = \sum_E \exp(-\beta E) \mathrm{Vol}(\Xi_E) \tag{2.18}$$

where

$$\mathrm{Vol}(\Xi_E) = \int_{\Xi_E} \mathrm{d}\mu_E \tag{2.19}$$

is the volume of the trajectories Ξ_E.

In [25], we explicitly express $\mathrm{d}\mu$ in terms of the solution space and the moduli of the MKdV equation. It is well known that any solution of the MKdV equation can be expressed by the hyperelliptic function and its moduli agree with those of the Jacobi varieties of the hyperelliptic curves [32,35]. Further the moduli of the hyperelliptic curves are a subset of the Siegel upper space [32, 35]. According to the arguments in [25], even though we introduce the infinite dimensional coordinates $\boldsymbol{t}$ in (2.12), they are often reduced to finite-dimensional space, as the Jacobi variety of a hyperelliptic curve with finite dimension is embedded in the universal Grassmannian manifold in Sato theory [25, 31, 32]. Using the genus g of the hyperelliptic curves, $(\Xi_E, \mathrm{d}\mu_E)$ can be decomposed as $(\Xi_E, \mathrm{d}\mu_E) = \coprod_g (\Xi_E^{(g)}, \mathrm{d}\mu_E^{(g)})$, where most $\Xi_E^{(g)}$ are empty sets for given E and their volume vanishes. In [25], we show that $\Xi_E^{(g)}$ is given as the real subspace of the Jacobi variety corresponding to the hyperelliptic curve with genus g, which is the trajectory space of the solution of the MKdV equation. For the case of a solution represented by the hyperelliptic function of genus g, $\mathrm{d}\mu_E^{(g)}$ is locally expressed as $\mathrm{d}t_3 \wedge \mathrm{d}t_5 \wedge \cdots \wedge \mathrm{d}t_{2g-1}$ where $\boldsymbol{t}_g := (t_1, t_3, \ldots, t_{2g-1})$ is a subset of the infinite-dimensional deformation parameters $\boldsymbol{t}$ in (2.12). Hence (2.19) becomes

$$\mathrm{Vol}(\Xi_E) = \sum_g \mathrm{Vol}(\Xi_E^{(g)}) = \sum_g \int_{\Xi_E^{(g)}} \mathrm{d}\mu_E^{(g)}. \tag{2.20}$$

Thus the volume of $\Xi_E^{(g)}$ is estimated by the unit of the elastica length L; due to the complex structure of the Jacobi variety of the hyperelliptic curve and that it admits the coordinate transformations such as rotation, the volume can be evaluated in terms of the elastica length L [25, 32, 35].

However, since the dimension of the non-empty trajectory space $\Xi_E^{(g)}$ is g, a sum of the terms with different dimensional volume appears in (2.18). It seems to be fancy but noting the facts that the dimension of the energy functional E is the inverse of the length and β/[length] is of order unity, the multiple of the length can be interpreted as the multiple of the quantization parameter β^{-1}. Hence such summation has a physical meaning.

The space of the formal direct sum set $\coprod_E \Xi_E$ is acted by the infinite-dimensional Lie group associated with the infinite-dimensional Lie algebra $A_1^{(1)}$ with restrictions due to the boundary condition (2.16) [31]. The space $\coprod_E \Xi_E$ can be regarded as a group manifold and the measure $\mathrm{d}\mu$ in the space can be interpreted as the Haar measure over the group manifold. In other words, the functional integral (2.5) reduces to the integral over the group manifold. Using natural topology induced from the group action, the distance in abstract space is naturally defined and parametrized by β. Due to such a group structure, the division by the affine group in (2.6) is justified.

Furthermore, deformation of the trajectory space $\Xi_E^{(g)}$ is equivalent to deformation of its corresponding Jacobi variety and implies a change of the energy E. Since the Jacobi variety of the hyperelliptic curve is classified by its modulus, or a point of the Siegel upper half plane [35], the deformation is expressed by an orbit in the moduli of the Jacobi varieties. A shape of the quantized elastica can be determined as a point of the trajectory space $\Xi_E^{(g)}$ of a solution of the MKdV equation after fixing the solution, or a point of the moduli, of the MKdV equation [25]. Hence for any energy interval $(E, E + \delta E)$, by using the theorem on implicit function and noting the boundary condition (2.16), one may evaluate the distribution of the volume of trajectory spaces over E. For sufficiently small δE, the distribution is expressed by the product of $\mathrm{Vol}(\Xi_E^{(g)})$ and the volume of a subset of the moduli of the Jacobi variety $\Omega_g(E)$ corresponding to $(E, E + \mathrm{d}E)$

$$\mathcal{Z}_{\mathrm{reg}} = \int \mathrm{d}E \left(\sum_g \mathrm{Vol}(\Xi_E^{(g)}) \Omega_g(E) \right) \exp(-\beta E). \tag{2.21}$$

Here the subset is given as the restriction of the moduli owing to the boundary condition (2.16) [25]. Hence (2.18) and (2.21) mean that the density of states of the Bernoulli–Euler functional system is completely represented by the moduli and solution spaces of the MKdV equation.

3. Quantization of Willmore surface

As the quantization procedure of an elastica has been reviewed, in this section, we investigate the quantization of a Willmore surface along the lines of the argument described in the previous section.

We denote by $\mathcal{S}$ a shape of a compact conformal surface immersed in the three-dimensional space $\mathbb{R}^3 \approx \mathbb{C} \times \mathbb{R}$, and by $(Z(z, \bar{z}) := X^1 + \mathrm{i}X^2, X^3(z, \bar{z}))$ its affine vector

$$\Sigma \ni z \mapsto (Z, X^3) \in \mathcal{S} \subset \mathbb{C} \times \mathbb{R}. \tag{3.1}$$

Here Σ can be expressed as $\Sigma = \mathbb{C}/\Gamma$ where Γ is a Fuchsian group and then Σ is a complex analytic object [33]. The volume element and the infinitesimal length of the surface $\mathcal{S}$ induced from $\mathbb{R}^3$ are given by

$$\mathrm{dvol} = \frac{\mathrm{i}}{2} \rho \, \mathrm{d}z \wedge \mathrm{d}\bar{z} =: \frac{\mathrm{i}}{2} \rho \, \mathrm{d}^2 z \qquad \mathrm{d}s^2 = \rho \, \mathrm{d}z \, \mathrm{d}\bar{z}. \tag{3.2}$$

The Konopelchenko–Kenmotsu–Weierstrass–Enneper (KKWE) relation is then expressed as

$$\psi_+ = \mathrm{i}\sqrt{\bar{\partial} \bar{Z}} \qquad \psi_- = -\mathrm{i}\sqrt{\partial \bar{Z}} \qquad \partial X^3 = -\bar{\psi}_+ \bar{\psi}_- \tag{3.3}$$

$$\mathcal{D}\psi = \begin{pmatrix} \partial & -V \\ V & \bar{\partial} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0 \qquad V = \frac{1}{2} H \rho \tag{3.4}$$

where $\partial := \partial/\partial z$ ($\bar{\partial} := \partial/\partial \bar{z}$), V is a real valued function of z and $\bar{z}$ and H is the mean curvature of $\mathcal{S}$ [10–14]. More general form of (3.4) for a non-conformal surface was calculated by Burgess and Jensen [18] following a prescription of the confinement Dirac operator [1]. Their equation in [18] is complicated but if one requires the conformal structure for the surface, the equation becomes simpler and given by (3.4) [16] as the isometry condition makes the Frenet–Serret relation simple like (2.2).

The energy integral of the Willmore surface or the Polyakov extrinsic curvature action is given as

$$E = \int \rho \, \mathrm{d}^2 z \, H^2 = 4 \int \mathrm{d}^2 z \, V^2 \tag{3.5}$$

the Willmore surface is realized as its stationary point.

In the same way that the elastica is quantized using the MKdV equation [25], we consider quantization of such a surface. The partition function of the surface is also given as [25]

$$\tilde{\mathcal{Z}}_{\rm reg} = \frac{\int DX \exp(-\beta \int \rho\, \mathrm{d}^2 z\, H^2)}{\mathrm{Vol(Aff)Vol(Conf)}} \tag{3.6}$$

where Vol(Aff) and Vol(Conf) mean the volume of the group of the affine transformations, and the conformal reparametrization of $\Sigma \to \Sigma$, respectively. The Willmore functional is also invariant for the translation of the centroid of the surface, special orthogonal rotation SO(3) and the reparametrization of Σ. The measure in (3.6) might be the Haar measure of a symmetry group of this system. Dividing them as a group by these trivial transformations groups, we bring our attention to bear on the shape of the surface in $\mathbb{R}^3$.

We search for the deformation flow of the surface which preserves the Willmore functional or the Polyakov extrinsic curvature action and conformal structure. Our question is what equation the deformation flow obeys. Taimanov and Konopelchenko produced an answer to the question; the modified Novikov–Veselov (MNV) equation preserves the conformal structure of $\mathcal{S}$ and the functional (3.5) [10–14].

The MNV equation is given as

$$V_t = V_{t^+} + V_{t^-} \qquad V_{t^+} = \partial^3 V + 3\partial V U + \tfrac{3}{2} V \partial U \qquad V_{t^-} = \bar\partial^3 V + 3\bar\partial V \bar U + \tfrac{3}{2} V \bar\partial \bar U$$
$$\bar\partial U = \partial V^2 \qquad \partial \bar U = \bar\partial V^2. \tag{3.7}$$

Along the curve $z = \bar z$, the MNV equation (3.7) is reduced to the MKdV equation (2.13).

As the Frenet–Serret relation can be regarded as the inverse scattering system of the MKdV equation, the KKWE equation can also be regarded as the inverse scattering system of the MNV equation.

$$(\partial_{t^\pm} - B^\pm)\not{D} + [\not{D}, A^\pm] = 0 \tag{3.8}$$

recovers (3.7) for

$$A^+ = \begin{pmatrix} \partial^3 & -3(\partial V)\partial + 3VU \\ 0 & \partial^3 + 3U\partial + 3(\partial U)/2 \end{pmatrix}$$
$$B^+ = 3\begin{pmatrix} 0 & (\partial V)\partial - VU \\ -(\partial V)\partial - (\partial^2 V) - UV & 0 \end{pmatrix}$$
$$A^- = \begin{pmatrix} \bar\partial^3 + \bar U\bar\partial + 3\bar\partial\bar U/2 & 0 \\ 3\bar\partial V\bar\partial - 3V\bar U & \bar\partial^3 \end{pmatrix}$$
$$B^- = 3\begin{pmatrix} 0 & (\bar\partial V)\bar\partial + (\bar\partial^2 V) - V\bar U \\ -(\bar\partial V)\bar\partial + V\bar U & 0 \end{pmatrix}. \tag{3.9}$$

The variation of the Dirac field is given as

$$\partial_t \psi = \partial_{t^+}\psi + \partial_{t^-}\psi \qquad \partial_{t^\pm}\psi = A^\pm \psi. \tag{3.10}$$

For the variation of t obeying the MNV equation, the Willmore functional is invariant

$$4\partial_t \int \mathrm{d}^2 z\, V^2 = \partial_t E = 0 \tag{3.11}$$

because the integrand can be expressed by the boundary quantities [13]

$$V_t^2 = \partial(V\partial^2 V - \tfrac{1}{2}(\partial V)^2 + \tfrac{3}{2}V^2 U) + \bar\partial(V\bar\partial^2 V - \tfrac{1}{2}(\bar\partial V)^2 + \tfrac{3}{2}V^2\bar U). \tag{3.12}$$

Next we check whether the conformal structure of the surface for the MNV flow is preserved following the argument of Taimanov.

First, we remark that the metric is represented by the Dirac field as

$$\rho = (|\psi_1|^2 + |\psi_2|^2)^2 \tag{3.13}$$

owing to the relation (3.3). Thus, if the relation (3.3) is covariant or preserves for the MNV flow, the conformal structure (3.13) is maintained.

Thus, we evaluate $\partial_t Z = \partial_{t^+} Z + \partial_{t^-} Z$ and $\partial_t X^3$. By straightforward computations, these values are calculated as [13]

$$\partial_{t^\pm} Z = 2\mathrm{i} \int^{z(\bar{z})} \mathrm{d}(f_\pm + g_\pm) \tag{3.14}$$

and

$$\partial_t X^3 = -\int^{z(\bar{z})} \mathrm{d}(h_1 + h_2) \tag{3.15}$$

where

$$\begin{aligned} f_+ &:= \tfrac{3}{2} U \psi_-^2 \qquad g_+ := \psi_- \partial^2 \psi_- - \tfrac{1}{2}(\partial \psi_-)^2 \\ f_- &:= \tfrac{3}{2} \bar{U} \psi_+^2 \qquad g_- := \psi_+ \bar{\partial}^2 \psi_+ - \tfrac{1}{2}(\bar{\partial} \psi_+)^2 \end{aligned} \tag{3.16}$$

$$\begin{aligned} h_1 &= \bar{\psi}_+ \partial^2 \psi_- + \psi_- \partial^2 \bar{\psi}_+ - \partial \psi_- \partial \bar{\psi}_+ + 3U \bar{\psi}_+ \psi_- \\ h_2 &= \psi_+ \bar{\partial}^2 \bar{\psi}_- + \bar{\psi}_- \bar{\partial}^2 \psi_+ - \bar{\partial} \bar{\psi}_- \bar{\partial} \psi_+ + 3\bar{U} \psi_+ \bar{\psi}_- \end{aligned} \tag{3.17}$$

$U^{(+)} = U$ and $U^{(-)} = \bar{U}$. Here $\mathrm{d}f = \partial f \, \mathrm{d}z + \bar{\partial} f \, \mathrm{d}\bar{z}$.

Let the infinitesimal flow obeying the MNV equation (3.7) module ϵ^2 be denoted as

$$(Z_\epsilon, X_\epsilon^3) := (Z, X^3) + \epsilon \partial_t (Z, X^3) + \mathcal{O}(\epsilon^2). \tag{3.18}$$

(3.14)–(3.17) mean that the infinitesimal variation is given as the integral of the closed form defined over Σ [34] and can be regarded as a single function of Σ. Since (Z, X^3) is also a periodic function of Σ, $(Z_\epsilon, X_\epsilon^3)$ is globally defined over Σ as a function of Σ.

On the other hand, (3.14)–(3.17) guarantee that $[\partial, \bar{\partial}] X_\epsilon^i = 0$, which means that we can locally define the independent coordinates z and $\bar{z}$ for the X_ϵ^i surface; we can locally find a complex coordinate system of an open set of Σ. Furthermore, due to the global properties, their coordinate system can be extended to the global coordinate and the connection of each open set is trivial due to $[\partial, \bar{\partial}] X_\epsilon^i = 0$ for any point of Σ.

Hence, the MNV flow preserves the conformal structure of the surface $\mathcal{S}$.

We wish to emphasize that the MNV problem is also an initial value problem: for any shape of compact conformal surface, the 'time' t of development of the surface obeying the MNVV equation (3.7) can be expressed and this flow conserves the energy functional and conformal structure. Hence the trajectory of the deformation parameter t give states of the same energy and its volume is the degeneracy of each energy E.

As in the quantization of elastica, the measure of the functional integral $\mathrm{d}\mu$ can be decomposed

$$\mathrm{d}\tilde{\mu} = \sum_E \mathrm{d}\tilde{\mu}_E \tag{3.19}$$

and the moduli of $\tilde{\mathcal{Z}}_{\mathrm{reg}}$ restricted by E denoted as $\tilde{\Xi}_E$

$$\tilde{\mathcal{Z}}_{\mathrm{reg}} = \int \mathrm{d}\tilde{\mu} \exp(-\beta E) = \sum_E \exp(-\beta E) \int_{\tilde{\Xi}_E} \mathrm{d}\tilde{\mu}_E = \sum_E \exp(-\beta E) \mathrm{Vol}(\tilde{\Xi}_E). \tag{3.20}$$

As for the case of the Bernoulli–Euler functional, the density of the states of the Willmore functional system might be completely represented by the moduli of the MNV equation.

However, the Willmore surface $\mathcal{S}$ in $\mathbb{R}^3$ has no natural measure of length from the point of view of the value of the Willmore functional (3.5) because for a global scale transformation in $\mathbb{R}^3$ $z \to \lambda z$ ($\lambda > 0$), the mean curvature changes as $H \to H/\lambda$ and the Willmore surface is invariant. Hence for a given energy E, there are infinite degenerate states related to the global scaling parameter $\lambda \in (0, \infty)$ and the regularized partition function $\tilde{\mathcal{Z}}_{\mathrm{reg}}$ also diverges.

However, along the curve of $z = \bar{z}$, the MNV flow obeys the MKdV equation which conserves the local length of the curve. Thus, using the conformal reparametrization, we adjust the parametrization of z so that the curve is in a flat plane; we suppress conformal freedom of the surface by fixing such parametrization z. On the MNV flow, the length of the curve is also a conserved quantity and is well defined. Due to the compactness of the surface $\mathcal{S}$, L must be finite. Hence in terms of this length, I can redefine the partition function by fixing the length of the curve

$$\mathcal{Z}_{\mathrm{reg}} := \tilde{\mathcal{Z}}_{\mathrm{reg}}|_{(\text{the length of } z=\bar{z})=L}. \tag{3.21}$$

After we fix the scale of the surface $\mathcal{S}$, we can define the decomposed measure and the space of trajectories by the restriction, $\mathrm{d}\mu_{E,L} := \mathrm{d}\tilde{\mu}_E|_L$, $\Xi_{E,L} := \tilde{\Xi}_L|_L$

$$\mathcal{Z}_{\mathrm{reg}} = \sum_E \int_{\Xi_{E,L}} \mathrm{d}\tilde{\mu}_{E,L} \exp(-\beta E) = \sum_E \exp(-\beta E)\mathrm{Vol}(\Xi_{E,L}). \tag{3.22}$$

The physical meaning of the summation in (3.22) is justified similarly to (2.21).

Since the MKdV equation is embedded in the MNV equation as the equation of motion of its equator $z = \bar{z}$, the space of the trajectories of the MNV equation can be classified by the genus g of the MKdV equation. Furthermore, the moduli of the MNV equation might be expressed by those of the Jacobi variety, at least, as the moduli of the equation of its equator by the restriction. Thus a shape of the Willmore surface can be determined as a point of the trajectory space of a solution of the MNV equation after fixing the solution or a point of the moduli of the MNV equation. By knowledge of the moduli of the MNV equation, one can evaluate the density of states of the Willmore surface using the theorem of implicit function like (2.21). Equation (3.22) can be expressed as

$$\mathcal{Z}_{\mathrm{reg}} = \int \mathrm{d}E \left(\sum_g \mathrm{Vol}(\Xi^{(g)}_{E,L})\Omega_g(E) \right) \exp(-\beta E) \tag{3.23}$$

where $\Omega_g(E)$ is the volume of the subset of the moduli of the Jacobi variety of the MNV equation in which the Willmore surfaces are satisfied with the boundary condition and have energy belonging to $(E, E + \mathrm{d}E)$.

Thus it means that quantization of the Willmore surface can be done using the structure of the moduli of the MNV equation.

4. Discussion

Carroll and Konopelchenko proved that the MNV flow conserves the extrinsic string action for the case where ρH is constant; here the extrinsic string action consists of the Nambu–Goto action, the Wess–Zumino–Witten type geometrical action, and the Polyakov extrinsic curvature action (3.3). Hence our result (3.2) can be extended to such a case and then it means that the algorithm of the calculation of the partition function of the extrinsic string in $\mathbb{R}^3$ is essentially the same as in earlier arguments. In other words, the quantization of the string immersed in $\mathbb{R}^3$ can be partially performed although only string in $\mathbb{R}^n$ $n < 3$ has been studied as a two-dimensional gravity [28–30].

In [2] and [3], it is stated that as the self-dual Yang–Mills equation can be expressed by an integrable equation and that its solutions can be represented by the Dirac operator over an associated principal bundle and as the MKdV equation governs the 'virtual' motion of an elastica and can be written by the Dirac operator over the elastica, the higher dimensional soliton surface might be expressed by the Dirac operator and hence the Dirac field would have a physical meaning.

This conjecture is supported by the discoveries and studies of Konopelchenko and Taimanov. Using the Dirac operator, they investigated the surface itself and derived non-trivial results [10–14]. The linear analytic system of Dirac operator [10–17], the differential geometry [17, 21, 22] and integrable system [12, 13] are closely connected with each other also in the two-dimensional system.

For the case of an elastica problem, both partition functions of the quantized Dirac field defined over the elastica [6] and the quantized elastica itself [25] are described by the MKdV hierarchy and the subalgebra of the affine Lie algebra $A_1^{(1)}$ [25, 31]. Behind them, Jimbo–Miwa theory and Sato theory exist and using them one could express and unify theories related to the elastica (and, of course, all one-dimensional solitons) [31]; it is expected that using Sato theory, natural chain complexes in these theories related to the elastica and MKdV hierarchy are interpreted as chain complex in the group acting on the universal Grassmannian manifold (UGM). (However, it is also noted that they are substantially formal theories and efficient only for a pinched Riemann surface; it does not lead us to find the concrete solutions of the 'virtual' dynamics of the closed elastica associated with the Jacobi variety of genus $g > 1$.)

Even though it is formal, the connections among the Dirac field, geometry and integrable equation related to two-dimensional objects also should be interpreted as functors among them and the associated morphisms in individual categories should be expressed by common language and unified. In other words, it is important to search for such a hidden symmetry like the group acting on the UGM for one-dimensional solitons and derive a theory to unify them. We believe that this quantization of the surface and the quantization of elastica [25] contribute to such studies.

Finally, we comment upon open problems related to this system. In [25], it was found that at the critical point of the quantized elastica, a certain expectation value obeys the Painlevé equation of the first kinds. We question what equation appears at the critical point in the quantized Willmore surface system. If one exists, it might be related to the higher dimensional analogue of the Painlevé equation.

Furthermore, since the MNV equation is an initial value problem, more general Riemannian surfaces can be allowed, at least as an initial condition, even though the energy manifold of the inverse scattering system of the MKdV equation, which is embedded in the MNV equation, is given as only hyperelliptic curves [35]. Hence, one may ask whether there is an analytical connection between the general Riemannian surface or the general Fuchsian group and the hyperelliptic function of the MKdV equation. If there is, this system should be algebraically studied.

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